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# COMPLEX NUMBER

*THEORY AND EXERCISE BOOKLET*

## CONTENTS




394 - Rajeev Gandhi Nagar Kota, Ph. No. 0744-2209671, 93141-87482, 93527-21564

IVRS No. 0744-2439051, 0744-2439052, 0744-2439053 www.motioniitjee.com, email-info@motioniitjee.com

## JEE Syllabus :

Algebra of complex numbers, addition, multiplication, conjugation, polar representation, properties of modulus and principal argument, triangle inequality, cube roots of unity, geometric interpretations.

## A. DEFINITION

Complex numbers are defined as expressions of the form a + ib where a , b $\in$ R & i = $\sqrt{-1}$. It is denoted by z i.e. z = a + ib . 'a' is called as real part of z (Re z) and 'b' is called as imaginary part of z (Im z).

**EVERY COMPLEX NUMBER CAN BE REGARDED AS**

| Purely real | Purely imaginary | Imaginary |
|---|---|---|
| if b = 0 | if a = 0 | if b $\neq$ 0 |

**Remark :**

**(a)** The set R of real numbers is a proper subset of the complex numbers . Hence the complete number system is N $\subset$ W $\subset$ I $\subset$ Q $\subset$ R $\subset$ C.

**(b)** Zero is both purely real as well as purely imaginary but not imaginary .

**(c)** i = $\sqrt{-1}$ is called the imaginary unit . Also $i^2 = -l$ ; $i^3 = -i$ ; $i^4 = 1$ etc.

**(d)** $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non – negative.

## B. ALGEBRAIC OPERATIONS

The algebraic operations on complex numbers are similar to those on real numbers treating 'i' as a polynomial . Inequalities in complex numbers are not defined . There is no validity if we say that complex number is positive or negative.

e.g. z > 0 , 4 + 2i < 2 + 4 i are meaningless.

However in real numbers if $a^2 + b^2 = 0$ then a = 0 = b but in complex numbers,

$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

**EQUALITY IN COMPLEX NUMBER :** Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts coincide.

## C. CONJUGATE COMPLEX

If z = a + ib then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$ . i.e. $\bar{z} = a - ib$.

**Remark :**

**(i)** $z + \bar{z} = 2\,\text{Re}(z)$ **(ii)** $z - \bar{z} = 2i\,\text{Im}(z)$ **(iii)** $z\bar{z} = a^2 + b^2$ which is real

**(iv)** If z lies in the 1st quadrant then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.

**Ex.1** Express $(1 + 2i)^2/(2 + i)^2$ in the form x + iy.

**Sol.** $\dfrac{(1+2i)^2}{(2+i)^2} = \dfrac{1+4i-4}{4+4i-1} = \dfrac{-3+4i}{3+4i} = \dfrac{(-3+4i)(3-4i)}{(3+4i)(3-4i)}$ $\therefore$ the expression $= \dfrac{-9+16+24i}{9+16} = \dfrac{7}{25} + i\dfrac{24}{25}$

**Ex.2** Show that a real value of x will satisfy the equation $\frac{1-ix}{1+ix} = a - ib$, if $a^2 + b^2 = 1$.

**Sol.** We have $\frac{1-ix}{1+ix} = a - ib$ or $ix = \frac{1-(a-ib)}{1+(a-ib)}$ [by componendo and dividendo],

or $x = \frac{1-a+ib}{b+i(1+a)} = \frac{\{(1-a)+ib\}\{b-i(1+a)\}}{b^2+(1+a)^2} = \frac{2b+i(a^2+b^2-1)}{b^2+(1+a)^2}$

Therefore, x will be real, if $a^2 + b^2 = 1$.

**Ex.3** Find the square root of a + ib

**Sol.** Let $\sqrt{a+ib} = x + iy$, where x and y are real. Squaring, $a + ib = x^2 - y^2 + i2xy$.

Equating real and imaginary parts, $a = x^2 - y^2$ ...(i) , $b = 2xy$ ...(ii)

Now $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = a^2 + b^2$ or $x^2 + y^2 = \sqrt{a^2+b^2}$ ...(iii)

[$\because$ x and y are real, the sum of their squares must be positive]

From (i) and (iii), $x^2 = \frac{\sqrt{a^2+b^2}+a}{2}$ or $x = \pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}$

and $y^2 = \frac{\sqrt{a^2+b^2}-a}{2}$ or $y = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$

If b is positive, both x and y have the same signs and in opposite case, contrary signs. [by (ii)].

## D. IMPORTANT PROPERTIES OF CONJUGATE / MODULUS / ARGUMENT

If $z, z_1, z_2 \in C$ then ;

**(a)** $z + \bar{z} = 2\,\text{Re}(z)$ ; $z - \bar{z} = 2i\,\text{Im}(z)$ ; $\overline{(\bar{z})} = z$ ; $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ ;

$\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$ ; $\overline{z_1z_2} = \bar{z}_1 . \bar{z}_2$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ ; $z_2 \neq 0$

**(b)** $|z| \geq 0$ ; $|z| \geq \text{Re}(z)$ ; $|z| \geq \text{Im}(z)$ ; $|z| = |\bar{z}| = |-z|$ ; $z\bar{z} = |z|^2$ ;

$|z_1 z_2| = |z_1|.|z_2|$ ; $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ , $z_2 \neq 0$ , $|z^n| = |z|^n$ ;

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

**(c) (i)** $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi$ . $k \in I$

**(ii)** $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi$ ; $k \in I$

**(iii)** $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$ .

where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

**Ex.4** The maximum & minimum values of $|z + 1|$ when $|z + 3| \le 3$ are

**Sol.** $|z + 3| \le 3$ denotes set of points on or inside a circle with centre $(-3, 0)$ and radius 3.
$|z + 1|$ denotes the distance of P from A $\Rightarrow$ $|z + 1|_{min} = 0$ & $|z + 1|_{max} = 5$

**Ex.5** Let $z_1$, $z_2$ be two complex numbers represented by points on the circle $|z_1| = 1$ and $|z_2| = 2$ respectively, then

**Sol.** $|2z_1 + z_2| \le 2|z_1| + |z_2| = 2 \times 1 + 2 = 4$

$\therefore$ Maximum value of $|2z_1 + z_2| = 4$ Clearly $|z_1 - z_2|$ is least when 0, $z_1$, $z_2$ are collinear.

Then $|z_1 - z_2| = 1$. Again $\left|z_2 + \frac{1}{z_1}\right| \le |z_2| + \left|\frac{1}{z_1}\right| = 2 + \frac{1}{|z_1|} = 2 + \frac{1}{1} = 3 \Rightarrow \left|z_2 + \frac{1}{z_1}\right| \le 3$

**Ex.6** Prove that if $z_1$ and $z_2$ are two complex numbers and $c > 0$, then
$|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$.

**Sol.** $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2$

$= |z_1|^2 + |z_2|^2 + 2R(z_1\bar{z}_2) \le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$

i.e. $|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + |z_1|\,|z_2|$. $[|\bar{z}_2| = |z_2|]$

Incorporating the number $c > 0$, the last term on the RHS can be written

$2|z_1|\,|z_2| = 2|\sqrt{c}z_1|\left|\frac{z_2}{\sqrt{c}}\right| \le |\sqrt{c}z_1|^2 + \left|\frac{z_2}{\sqrt{c}}\right|^2$ i.e., $2|z_1|\,|z_2| \le c|z_1|^2 + \frac{|z_2|^2}{c}$ $[2ab \le a^2 + b^2]$

Hence $|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + c|z_1|^2 + \frac{|z_2|^2}{c} = (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$

**Ex.7** If $z_1$, $z_2$, $z_3$ are the points A, B, C in the Argand Plane such that,
$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$, prove that ABC is an equilateral triangle.

**Sol.** Let $z_2 - z_3 = p$ ; $z_3 - z_1 = q$ ; $z_1 - z_2 = r \Rightarrow p + q + r = 0$
Given condition, $pq + qr + rp = 0 \Rightarrow p(q + r) + qr = 0 \Rightarrow p(-p) + qr = 0$

$\Rightarrow p^2 = qr \Rightarrow (\bar{p})^2 = \bar{q}\,\bar{r} \Rightarrow (p\bar{p})^2 = (q\,r)\,(\overline{q\,r}) \Rightarrow (p\bar{p})^3 = (p\bar{p})\,(q\,r)\,(\overline{q\,r})$

$\Rightarrow (p\bar{p})^3 = (p\,q\,r)\,(\overline{p\,q\,r})$. Similarly others. Hence $p\bar{p} = q\bar{q} = r\bar{r} \Rightarrow |p| = |q| = |r|$

**Ex.8** If z & w are two complex numbers simultaneously satisfying the equations, $z^3 + w^5 = 0$ and $z^2 . \bar{w}^4 = 1$, then

**Sol.** $z^3 = -w^5 \Rightarrow |z|^3 = |w|^5 \Rightarrow |z|^6 = |w|^{10}$ .....(1) & $z^2 = \frac{1}{\bar{w}^4} \Rightarrow |z|^2 = \frac{1}{|w|^4} \Rightarrow |z|^6 = \frac{1}{|w|^{12}}$ .....(2)

From (1) & (2) $|w| = 1$ & $|z| = 1 \Rightarrow z\bar{z} = w\bar{w} = 1$ Again $z^6 = w^{10}$ ......(3)

and $z^6 . \bar{w}^{12} = 1 \Rightarrow z^6 = \frac{1}{\bar{w}^{12}} = w^{10}$ (from 3) $\Rightarrow (w\bar{w})^{10}(\bar{w})^2 = 1 \Rightarrow (\bar{w})^2 = 1$

$\Rightarrow \bar{w} = 1$ or $-1 \Rightarrow w = 1$ or $-1$
if $w = 1$ then $z^3 + 1 = 0$ and $z^2 = 1$ $\Rightarrow z = -1$
if $w = -1$ then $z^3 - 1 = 0$ and $z^2 = 1$ $\Rightarrow z = 1$
Hence $z = 1$ & $w = -1$ or $z = -1$ & $w = 1$

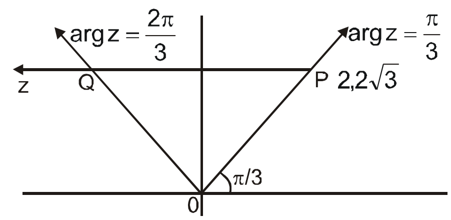
**Ex.9** The complex numbers whose real and imaginary parts are integers and satisfy the relation

$z\bar{z}^3 + z^3\bar{z} = 350$ forms a rectangle on the Argand plane, the length of whose diagonal is

**Sol.** $z\bar{z}\,(\bar{z}^2 + z^2) = 2\,(x^2 + y^2)\,(x^2 - y^2) = 350 \Rightarrow (x^2 - y^2)\,(x^2 + y^2) = 175 = 35.5 = 25.7$

$\Rightarrow x^2 + y^2 = 25\ \&\ x^2 - y^2 = 7 \Rightarrow x = \pm 4\ \&\ y = \pm 3$

**Ex.10** Find the area bounded by the curve, $\arg z = \frac{\pi}{3}$, $\arg z = \frac{2\pi}{3}$ and $\arg\left(z - 2 - 2\sqrt{3}i\right) = \pi$ in the complex plane .




**Sol.** required area, the equilateral triangle OPQ with side 4

$= \frac{\sqrt{3}}{4} . 16 = 4\sqrt{3}$

**Ex.11** Find the complex number where the curves $\arg(z - 3i) = 3\pi/4$ & $\arg(2z + 1 - 2i) = \pi/4$ intersect.

**Sol.** $\arg(z - 3i) = \frac{3\pi}{4} \Rightarrow \frac{y-3}{x} = \tan\frac{3\pi}{4} \Rightarrow x + y = 3$ & $\arg(2z + 1 - 2i) = \frac{\pi}{4}$

gives $\frac{2y-2}{2x+1} = \tan\frac{\pi}{4} \Rightarrow 2y - 2x = 3$ point of intersection is $\frac{3}{4} + \frac{9}{4}i$

**Ex.12** If $\left|\frac{z}{|\bar{z}|} - \bar{z}\right| = 1 + |z|$, then prove that z is a purely imaginary number.

**Sol.** Given that $\left|\frac{z}{|\bar{z}|} - \bar{z}\right| = 1 + |z|$ Put $z = re^{i\theta} \Rightarrow \bar{z} = re^{-i\theta}$

$\Rightarrow \left|\frac{z}{|\bar{z}|} - \bar{z}\right| = |e^{i\theta} - re^{-i\theta}| = 1 + r \Rightarrow (1 - r)^2\cos^2\theta + (1 + r)^2\sin^2\theta = (1 + r)^2$

$\Rightarrow (1 - r)^2\cos^2\theta - (1 + r)^2\cos^2\theta = 0 \Rightarrow \cos^2\theta = 0 \Rightarrow \mathrm{Re}(z) = 0$

$\Rightarrow$ z is a purely imaginary number.

**Ex.13** For $|z - 1| = 1$, show that $\tan\left(\frac{\arg(z-1)}{2}\right) - \frac{2i}{z} = -i$.

**Sol.** Here $z - 1 = e^{i\theta}$ so that $z = 1 + \cos\theta + i\sin\theta$

$\Rightarrow z = 2\cos^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \Rightarrow z = 2\cos\frac{\theta}{2}\,(^{i\theta/2})$.

Hence $\tan\left(\frac{\arg(z-1)}{2}\right) - \frac{2i}{z} = \tan\frac{\theta}{2} - \frac{i}{\cos\theta/2}e^{-i\theta/2} \Rightarrow \tan\frac{\theta}{2} - i\frac{\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}} = -i.$

**Ex.14** if $\left|\dfrac{2iz_1 - z_1 - z_2}{2iz_1 + z_1 + z_2}\right| = \left|\dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right|$, then prove that $\dfrac{z_1}{z_2}$ is purely real.

**Sol.** The given relation can be written as $\left|\dfrac{z_1 + i\dfrac{z_1+z_2}{2}}{z_1 - i\dfrac{z_1+z_2}{2}}\right| = 1 \Rightarrow \left|\dfrac{\dfrac{2z_1}{z_1+z_2} + i}{\dfrac{2z_1}{z_1+z_2} - i}\right| = 1$

$\Rightarrow$ $\dfrac{2z_1}{z_1+z_2}$ is real $\Rightarrow$ $\dfrac{2z_1}{z_1+z_2} = \dfrac{2\bar{z}_1}{\bar{z}_1+\bar{z}_2}$.

$\Rightarrow$ $z_1(\bar{z}_1 + \bar{z}_2) = \bar{z}_1(z_1 + z_2)$ $\Rightarrow$ $\dfrac{z_1}{z_2} = \dfrac{\bar{z}_1}{\bar{z}_2} \Rightarrow \dfrac{z_1}{z_2}$ is purely real.

**Ex.15** If a, b are complex and one of the roots of the equation $x^2 + ax + b = 0$ is purely real where as the other is purely imaginary, prove that $a^2 - \bar{a}^2 = 4b$.

**Sol.** Let $\alpha$ be the real and $i\beta$ be the imaginary roots of the given equation. Then

$\alpha + i\beta = -a \Rightarrow \alpha - i\beta = -\bar{a}$ $\Rightarrow$ $2\alpha = -(a + \bar{a})$ and $2i\beta = = -(a - \bar{a})$

so that $4i\alpha\beta = a^2 - \bar{a}^2 \Rightarrow 4b = a^2 - \bar{a}^2$

**Alternative solution :**

If one root is real and the other is imaginary, their product will be imaginary $\Rightarrow$ b is purely imaginary.

Let b = ik, so that the equation $x^2 + ax + ik = 0$ has one purely real root.

Let it be $\alpha \Rightarrow \alpha^2 + a\alpha + ik = 0 \Rightarrow \alpha^2 + \bar{a}\alpha - ik = 0$.

Hence $\dfrac{\alpha^2}{ika - i\bar{a}k} = \dfrac{\alpha}{ik + ik} = \dfrac{1}{\bar{a} - a}$ $\Rightarrow$ $\alpha^2 = \dfrac{ik(a+\bar{a})}{a-\bar{a}}$ and $\alpha = \dfrac{-2ik}{a-\bar{a}}$, so that

$\dfrac{ik(a+\bar{a})}{a-\bar{a}} = \dfrac{-4k^2}{(a-\bar{a})}$ $\Rightarrow$ $a^2 - \bar{a}^2 = 4ik = 4b$.

**Ex.16** For every real number $a \ge 0$, find all the complex numbers z that satisfy the equation
$2|z| - 4az + 1 + ia = 0$.

**Sol.** We have $2|z| - 4az + 1 + ia = 0$

Put z = x + i y, We get, $2\sqrt{x^2 + y^2} = 4ax - 1 + 4aiy - ia$ or $4(x^2 + y^2) = (4ax - 1)^2$ .....(1)

and a = 4 ay (by separating imaginary and real parts)

$\Rightarrow$ $y = \dfrac{1}{2}$ and $4x^2 + \dfrac{1}{4} - 16a^2x^2 - 1 + 8ax = 0$ $\Rightarrow$ $x^2(16 - 64a^2) + 32ax - 3 = 0$

$\Rightarrow$ $x = \dfrac{-4a \pm \sqrt{4a^2+3}}{4(1-4a^2)}$ as $x > \dfrac{1}{4a}$ (from equation (1))

$\Rightarrow$ either $\dfrac{4a + \sqrt{4a^2+3}}{16a^2 - 4} > \dfrac{1}{4a}$ $\Rightarrow$ $\dfrac{3}{4(4a - \sqrt{4a^2+3})} > \dfrac{1}{4a}$ $\Rightarrow$ $3a > 4a - \sqrt{4a^2+3}$

If a > $\frac{1}{2}$, $3(a^2 + 1) > 0$ is always true ; If a < $\frac{1}{2}$, $4a^2 + 3 < a^2$ is never true

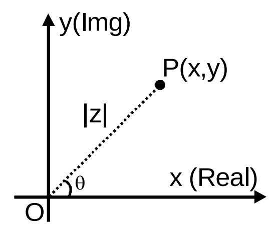
or, $\frac{4a-\sqrt{4a^2+3}}{16a^2-4} > \frac{1}{4a}$ $\Rightarrow$ $\frac{3}{4(4a+\sqrt{4a^2+3})} > \frac{1}{4a}$

$\Rightarrow$ a + $\sqrt{4a^2+3}$ < 0, which can never hold $\Rightarrow$ x = $\frac{4a+\sqrt{4a^2+3}}{16a^2-4} + \frac{i}{4}$ if a > $\frac{1}{2}$.

$\Rightarrow$ no solution if $0 \le a \le \frac{1}{2}$ and z = $\frac{4a+\sqrt{4a^2+3}}{16a^2-4} + \frac{1}{4}$ if a > $\frac{1}{2}$.

## E. REPRESENTATION OF A COMPLEX NUMBER

**(a) CARTESIAN FORM (GEOMETRIC REPRESENTATION) :** Every complex number $z = x + i\,y$ can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x , y) .




Length OP is called modulus of the complex number denoted by $|z|$ & $\theta$ is called the argument or amplitude.

eg. $|z| = \sqrt{x^2+y^2}$ &

$\theta = \tan^{-1} y/x$ (angle made by OP with positive x–axis)

**Remark :**

**(i)** $|z|$ is always non negative . Unlike real numbers $|z| = \begin{cases} z & \text{if } z>0 \\ -z & \text{if } z<0 \end{cases}$ is **not correct**

**(ii)** Argument of a complex number is a many valued function . If $\theta$ is the argument of a complex number then $2\,n\pi + \theta$ ; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$ .

**(iii)** The unique value of $\theta$ such that $-\pi < \theta \le \pi$ is called the principal value of the argument.

**(iv)** Unless otherwise stated, amp z implies principal value of the argument.

**(v)** By specifying the modulus & argument a complex number is defined completely. For the complex number 0 + 0 i the argument is not defined and this is the only complex number which is given by its modulus.

**(vi)** There exists a one–one correspondence between the points of the plane and the members of the set of complex numbers.

**(b) TRIGONOMETRIC / POLAR REPRESENTATION :**

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$ ; $\arg z = \theta$ ; $\bar{z} = r(\cos\theta - i\sin\theta)$

**Remark :** $\cos\theta + i\sin\theta$ is also written as CiS $\theta$ .

Also $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ & $\sin x = \frac{e^{ix}-e^{-ix}}{2}$ are known as Euler's identities.

**Ex.17** Express $z = \frac{-1+i\sqrt{3}}{1+i}$ in polar form and then find the modulus and argument of z. Hence deduce the value of $\cos\frac{5\pi}{12}$.

**Sol.** Let $-1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$. Equating real and imaginary parts, $r\cos\theta = -1$, $r\sin\theta = \sqrt{3}$.

Now $r^2 = 1 + 3 = 4$, $r = 2$, $\cos\theta = -\frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$ or $\theta = \frac{2\pi}{3}$ between $-\pi$ and $\pi$.

Consequently, $-1 + i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$. Similarly, $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

$$\therefore\ z = \frac{2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} = \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\left\{\left(\cos\frac{2\pi}{3}\cos\frac{\pi}{4} + \sin\frac{2\pi}{3}\sin\frac{\pi}{4}\right) + i\left(\sin\frac{2\pi}{3}\cos\frac{\pi}{4} - \cos\frac{2\pi}{3}\sin\frac{\pi}{4}\right)\right\} = \sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

It is the polar form of z. Obviously, $|z| = \sqrt{2}$ and $\arg z = \frac{5}{12}$ (principal value).

Again, $z = \frac{-1+i\sqrt{3}}{1+i} = \frac{(-1+i\sqrt{3})(1-i)}{1^2 - i^2} = \frac{1}{2}\{(\sqrt{3} - 1) + i(\sqrt{3} + 1)\}$.

$\therefore\ \sqrt{2}\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2}$ or $\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

**(c) EXPONENTIAL REPRESENTATION :** $z = re^{i\theta}$ ; $|z| = r$ ; $\arg z = \theta$ ; $\bar{z} = re^{-i\theta}$

**(d) VECTORIAL REPRESENTATION :** Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

**Remark :**

**(i)** If $\vec{OP} = z = re^{i\theta}$ then $\vec{OQ} = z_1 = re^{i(\theta+\phi)} = z \cdot e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\, e^{i\phi}$

**(ii)** If A, B, C & D are four points representing the complex numbers $z_1, z_2, z_3$ & $z_4$ then AB $||$ CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely real ; AB $\perp$ CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary

**(iii)** If $z_1, z_2, z_3$ are the vertices of an equilateral triangle where $z_0$ is its circumcentre then

**(a)** $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$ **(b)** $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

**Ex.18** If (1 – i) is a root of the equation, $z^3 - 2(2 - i)z^2 + (4 - 5i)z - 1 + 3i = 0$, then find the other two roots.

**Sol.** $z_1 + z_2 + z_3 = 2(2 - i) \Rightarrow z_2 + z_3 = 3 - i \; (\because z_1 = 1 - i)$ .......(1)

again $z_1 z_2 z_3 = 1 - 3i \Rightarrow z_2 z_3 = \frac{1-3i}{1-i} = 2 - i$ .......(2)

From (1) & (2) $z_2 = 1$ & $z_3 = 2 - i$

**Ex.19** Prove that if the ratio $\frac{z-i}{z-1}$ is purely imaginary then the point z lies on the circle whose centre is at the point $\frac{1}{2}(1 + i)$ and radius is $\frac{1}{\sqrt{2}}$.

**Sol.** Let z = x + iy.

Then $\frac{z-i}{z-1} = \frac{x+i(y-1)}{x-1+iy} = \frac{\{x+i(y-1)\}\{(x-1)-iy\}}{(x-1)^2+y^2} = \frac{x(x-1)+y(y-1)}{(x-1)^2+y^2} + i\frac{(x-1)(y-1)-xy}{(x-1)^2+y^2}$

Since $\frac{z-i}{z-1}$ is purely imaginary, $x(x - 1) + y(y - 1) = 0$ or $\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2}$

It is a circle with radius $\frac{1}{\sqrt{2}}$ and centre $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Therefore, the point z lies on a circle and the centre is $\frac{1}{2}(1 + i)$.

**Ex.20** A function *f* is defined on the complex number by *f* (z) = (a + b*i*)z, where 'a' and 'b' are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that | a + b*i* | = 8 and that $b^2 = \frac{u}{v}$ where *u* and *v* are co-primes. Find the value of (*u* + *v*).

**Sol.** Given $|(a + bi)z - z| = |(a + bi)z| \Rightarrow |z(a - 1) + biz| = |az + bzi|$

$\Rightarrow |z|\,|(a - 1) + bi| = |z|\,|a + bi| \quad \therefore \quad (a - 1)^2 + b^2 = a^2 + b^2 \quad \therefore \quad a = 1/2$

since $|a + bi| = 8 \Rightarrow a^2 + b^2 = 64 \Rightarrow b^2 = 64 - \frac{1}{4} = \frac{255}{4} = \frac{u}{v} \quad \therefore \; u = 255 \;\&\; v = 4 \;\Rightarrow\; u + v = 259$

**Ex.21** Show that $\tan^{-1}\frac{1}{5}$ is nearly equal to $\frac{\pi}{16}$.

**Sol.** We have $(5 + i) = \sqrt{26}(\cos\theta + i\sin\theta)$, where $\tan\theta = \frac{1}{5}$ and therefore $(5 + i)^4 = 676(\cos 4\theta + i\sin 4\theta)$.

But $(5 + i)^4 = (24 + 10i) = 476 + 480i$ ; hence we have

$\cos 4\theta = 476/676$, $\sin 4\theta = 480/676$, and $\tan 4\theta = 1$, nearly. $\therefore\; 4\theta = \pi/4$ approximately.

**Ex.22** Find all complex numbers z which satisfy the equation $\exp\left(\dfrac{|z|^2 - |z| + 4}{|z|^2 + 1} \cdot \ell n 2\right) = \log_{\sqrt{2}}\left(\left|3\sqrt{15} + 11i\right|\right)$ .

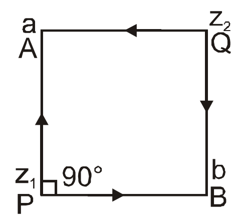
**Sol.** $e^{\frac{r^2 - r + 4}{r^2 + 1}\ell n 2} = \log_{\sqrt{2}} 16 = 8 \Rightarrow 2^{\frac{r^2 - r + 4}{r^2 + 1}} = 2^3 \Rightarrow r^2 - r + 4 = (r^2 + 1)\,3 \Rightarrow r = \dfrac{1}{2}$ or r = – 1 (rejected)

$|z| = \dfrac{1}{2}$ ; $z = \dfrac{1}{2}(\cos\theta + i\sin\theta)$

**Ex.23** If a & b are complex numbers then find the complex numbers $z_1$ & $z_2$ so that the points $z_1$ , $z_2$ and a , b be the corners of the diagonals of a square .

**Sol.** $a - z_1 = (b - z_1)\, e^{i\pi/2} = i\,(b - z_1)$

$a - i\,b = z_1\,(1 - i)$




$$z_1 = \frac{a - ib}{1 - i} = \frac{(a - ib)\,(1 + i)}{2} \quad \Rightarrow \quad z_1 = \frac{(a + b) + i\,(a - b)}{2}$$

Similarly $b - z_2 = (a - z_2)\, e^{i\pi/2} \quad \Rightarrow \quad z_2 = \dfrac{a+b}{2} - i\left(\dfrac{a-b}{2}\right)$

Alternately $|z_1 - a|^2 = |z_2 - b|^2 \Rightarrow (z_1 - a)(\bar{z}_1 - \bar{a}) = (z_1 - b)(\bar{z}_1 - \bar{b}) \Rightarrow \dfrac{\bar{z}_1 - \bar{b}}{\bar{z}_1 - \bar{a}} = \dfrac{z_1 - a}{z_1 - b}$ .....(1)

Again $\arg\dfrac{z_1 - b}{z_1 - a} = \dfrac{\pi}{2} \Rightarrow \dfrac{z_1 - b}{z_1 - a}$ is purely imaginary. Hence $\dfrac{z_1 - b}{z_1 - a} = -\dfrac{\bar{z}_1 - \bar{b}}{\bar{z}_1 - \bar{a}}$ ......(2)

From (1) & (2) , $\dfrac{z_1 - a}{z_1 - b} = -\dfrac{z_1 - b}{z_1 - a}$ or $(z_1 - a)^2 + (z_1 - b)^2 = 0$

or $2z_1^2 - 2z_1(a + b) + a^2 + b^2 = 0$ or $z_1 = \dfrac{(a + b) + i\,(a - b)}{2}$ & $z_2 = \dfrac{a+b}{2} - i\left(\dfrac{a-b}{2}\right)$

**Ex.24** Let a sequence $x_1, x_2, x_3, \ldots\ldots$ of complex numbers be defined by $x_1 = 0$, $x_{n+1} = x_n^2 - i$ for n > 1 where $i^2 = 1$. Find the distance of $x_{2000}$ from $x_{1997}$ in the complex plane.

**Sol.** $x_1 = 0$, $x_2 = 0^2 - i = -i$, $x_3 = (-i)^2 - i = -1 - i = -(1 + i)$,

$x_4 = [_1(1 + i)]^2 - i = 2i - i = i$, $x_5 = (i)^2 - i = -1 - i = x_3$, $x_6 = (-1 - i)^2 - i = i = x_4$

$\therefore$ $x_6 = x_4$ and hence $x_7 = x_5$ and so on $x_{2n} = i$ for $n \geq 1$, $x_{2n+1} = -1 - i$

So $x_{2000} = i = (0, 1)$ in the complex plane, $x_{1997} = (1, -i) = (-1, -1)$ in the complex plane.

So, distance between $x_{2000}$ and $x_{1997}$ is $\sqrt{1^2 + 2^2} = \sqrt{5}$

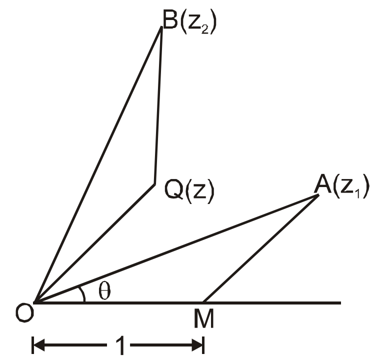
**Ex.25** Find the square root of $x + \left(\sqrt{x^4 + x^2 + 1}\right) i$ .

**Sol.** $E = x + \sqrt{\left(x^2 + x + 1\right)\left(x^2 - x + 1\right)}\; i = \dfrac{\left(x^2 + x + 1\right)}{2} + \dfrac{\left(x^2 - x + 1\right)}{2}\, i^2 + \sqrt{\left(x^2 + x + 1\right)\left(x^2 - x + 1\right)}\; i$

$$= \left(\sqrt{\frac{x^2 + x + 1}{2}} + \sqrt{\frac{x^2 - x + 1}{2}}\; i\right)^2 \quad \Rightarrow \quad \sqrt{E} = \pm\,\frac{1}{\sqrt{2}}\left(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x + 1}\; i\right)$$

**Ex.26** On the Argand plane point 'A' denotes a complex number $z_1$. A triangle OBQ is made directly similar to the triangle OAM, where OM = 1 as shown in the figure. If the point B denotes the complex number $z_2$, then find the complex number corresponding to the point 'Q' in terms of $z_1$ & $z_2$.

**Sol.** $\frac{OB}{OQ} = \frac{OA}{OM} = OA$ $\quad (\because\ OM = 1)$




$OQ = \frac{OB}{OA}$ or $|z| = \frac{|z_2|}{|z_1|}$

Also amp $\frac{\vec{OB}}{\vec{OA}}$ = amp $\vec{OB}$ – amp $\vec{OA}$

$= \angle BOM - \angle AOM = \angle BOM - \angle BOQ = \angle QOM$ = amp of z $(\angle AOM = \angle BOQ = \theta)$

Hence complex number corresponding to the point Q = $\frac{z_2}{z_1}$

**Ex.27** For every real number a > 0 find all complex numbers z satisfying the equation, $z|z| + az + 1 = 0$.

**Sol.** Equating real and imaginary points, $x\sqrt{x^2+y^2} + ax = 0$ ........(1)

& $\quad y\sqrt{x^2+y^2} + ay + 1 = 0$ .........(2)

equation (1) gives x = 0

$\Rightarrow y|y| + ay + 1 = 0 \Rightarrow y^2 + ay + 1 = 0$ if $y \geq 0$ & $-y^2 + ay + 1 = 0$ if $y < 0$

If $y \geq 0$ then first equation gives no solution as $a > 0$ & second equation gives unique solution

$$z = \left(\frac{a - \sqrt{a^2+4}}{2}\right)i$$

**Ex.28** Compute the product $\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right]......\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right]$, where $n \geq 2$

**Sol.** Assume $\frac{1+i}{2} = z$ ; multiply numerator and denominator by (1 – z) which simplifies to

$= \frac{1-\left(z^2\right)^{2^n}}{1-z}$ ; Now $\quad \frac{1}{1-z} = \frac{2}{1-i} = (1+i)\ \left(z^{2^n}\right)^2 = \left(z^2\right)^{2^n} = \left[\left(\frac{1+i}{2}\right)^2\right]^{2^n} = \left(\frac{i}{2}\right)^{2^n}$

for $n \geq 2$ $(i)^{2^n} = 1 \Rightarrow \left(z^{2^n}\right)^2 = \frac{1}{2^{2^n}} \Rightarrow$ Given expression = $\left(1 - \frac{1}{2^{2^n}}\right)(1+i)$

**Ex.29** Find the set of points on the complex plane such that $z^2 + z + 1$ is real and positive (where $z = x + iy$) .

**Sol.** $x^2 - y^2 + 2xyi + x + iy + 1$ is real and positive $\Rightarrow$ $(x^2 - y^2 + x + 1) + y(2x + 1)i$ is real and positive
$\Rightarrow$ $y(2x + 1) = 0$ and $x^2 - y^2 + x + 1 > 0$ if $y = 0$ then $x^2 + x + 1$ is always positive

$\Rightarrow$ complete x – axis if $x = -\dfrac{1}{2}$ then $\dfrac{3}{4} - y^2 > 0$ $\Rightarrow$ $y \in \left(-\dfrac{\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2}\right)$

**Ex.30** The altitudes from the vertices A, B and C of the triangle ABC meet its circumcircle at D, E and F respectively. The complex numbers representing the points D, E and F are $z_1$, $z_2$ and $z_3$ respectively. If $\dfrac{z_3 - z_1}{z_2 - z_1}$ is purely real then show that triangle ABC is right angled at A.

**Sol.** The angles of $\Delta$DEF are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$ respectively. Also it is given that $\dfrac{z_3 - z_1}{z_2 - z_1}$ is purely real

$\Rightarrow$ $\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = 0$ or $\pi \Rightarrow$ $\pi - 2A = 0$ or $\pi$ $\Rightarrow$ $A = \dfrac{\pi}{2}$ or 0 (not permissible)

Hence triangle ABC is right angled at A.

## F. DEMOIVRE'S THEOREM

**STATEMENT :** $\cos n\theta + i \sin n\theta$ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$, $n \in Q$. The theorem is very useful in determining the roots of any complex quantity .
**Remark :** Continued product of the roots of a complex quantity should be determined using theory of equations.

**Ex.31** Simplify : $\dfrac{(\cos 3\theta + i\sin 3\theta)^7\,(\cos 5\theta - i\sin 5\theta)^4}{(\cos 4\theta + i\sin 4\theta)^{10}\,(\cos 13\theta - i\sin 13\theta)^3}$

**Sol.** Given expression $= \dfrac{(\cos\theta + i\sin\theta)^{21}\,(\cos\theta + i\sin\theta)^{-20}}{(\cos\theta + i\sin\theta)^{40}\,(\cos\theta + i\sin\theta)^{-39}} = \dfrac{(\cos\theta + i\sin\theta)}{(\cos\theta + i\sin\theta)} = 1$

**Ex.32** If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, prove that

**(i)** $\Sigma\cos 3\alpha = 3\cos(\alpha + \beta + \gamma)$ **(ii)** $\Sigma\sin 3\alpha = 3\sin(\alpha + \beta + \gamma)$ **(iii)** $\Sigma\cos^2\alpha = \Sigma\sin^2\alpha = \dfrac{3}{2}$.

**Sol.** To prove (i) and (ii) we put
$x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\alpha$ and $z = \cos\gamma + i\sin\gamma$.
$x + y + z = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$
Now $x^3 + y^3 + z^3 = 3xyz$ ($\because$ $x + y + z = 0$)
$\therefore$ $(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3\,(\cos\gamma + i\sin\gamma)^3$
$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$,
or $(\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)$
$= 3\{\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)\}$.
Equating the real and the imaginary parts, we get
$\Sigma\cos 3\alpha = 3\cos(\alpha + \beta + \gamma)$ and $\Sigma\sin 3\alpha = 3\sin(\alpha + \beta + \gamma)$.

(iii) We have $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = x^{-1} + y^{-1} + z^{-1}$ or $\frac{xy+yz+zx}{xyz}$

$= (\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma)$
$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma) = 0,$
i.e. $xy + yz + yx = 0.$
Now $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + yx)$
$\therefore$ $x^2 + y^2 + z^2 = 0$ ($\because$ $x + y + z = 0$, $xy + yz + zx = 0$),
or $\Sigma\cos 2\alpha + i\,\Sigma\sin 2\alpha = 0$. It leads to $\Sigma\cos 2\alpha = 0$, i.e., $\Sigma\cos^2\alpha = \Sigma\sin^2\alpha = k$ (say).

$\therefore$ $2k = \Sigma\cos^2\alpha + \Sigma\sin^2\alpha = 3$, or $k = \frac{3}{2}$

**Ex.33** If $\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$ then prove that,

$\Sigma\cos 4\alpha = 2\,\Sigma\cos 2(\beta+\gamma)$ & $\Sigma\sin 4\alpha = 2\,\Sigma\sin 2(\beta+\gamma)$.

**Sol.** Let $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ — (1)
$\therefore$ $x + y + z = (\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma)$
$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i\,.\,0 = 0$ (as given)
Now $x + y + z = 0$ gives $x + y = -z$ or $(x + y)^2 = z^2$, (squaring both sides)
i.e. $x^2 + y^2 - z^2 = -2xy$ or $(x^2 + y^2 - z^2)^2 = 4x^2y^2$, (again squaring both sides)
or $x^4 + y^4 + z^4 + 2x^2y^2 - 2x^2z^2 - 2y^2z^2 = 4x^2y^2$ or $x^4 + y^4 + z^4 = 2(x^2y^2 + y^2z^2 + z^2x^2)$
or $\Sigma x^4 = 2\,\Sigma y^2z^2$, (expressing in the summation notation)
or $\Sigma(\cos\alpha + i\sin\alpha)^4 = 2\,\Sigma(\cos\beta + i\sin\beta)^2(\cos\gamma + i\sin\gamma)^2$, putting for x, y & z from (1)
or $\Sigma(\cos 4\alpha + i\sin 4\alpha) = 2\,\Sigma(\cos 2\beta + i\sin 2\beta)(\cos 2\gamma + i\sin 2\gamma) = 2\,\Sigma[\cos 2(\beta+\gamma) + i\sin 2(\beta+\gamma)]$
Equating real and imaginary parts on both sides, we get
$\Sigma\cos 4\alpha = 2\,\Sigma\cos 2(\beta+\gamma)$ and $\Sigma\sin 4\alpha = 2\,\Sigma\sin 2(\beta+\gamma)$

**Ex.34** If $\alpha$, $\beta$ be the roots of the equation $u^2 - 2u + 2 = 0$ & if $\cot\theta = x + 1$, then $\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta}$ is equal to

**Sol.** $u^2 - 2u + 2 = 0 \Rightarrow u = 1 \pm i$

LHS $= \frac{[(\cot\theta - 1) + (1+i)]^n - [(\cot\theta - 1) + (1-i)]^n}{2i} = \frac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{\sin^n\theta\ 2i} = \frac{2i\sin n\theta}{\sin^n\theta\ 2i} = \frac{\sin n\theta}{\sin^n\theta}$

## G. CUBE ROOT OF UNITY

**(i)** The cube roots of unity are $1$, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$.
**(ii)** If $\omega$ is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.
**(iii)** In polar form the cube roots of unity are : $\cos 0 + i\sin 0$ ; $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$
**(iv)** The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.
**(v)** The following factorization should be remembered : ($a, b, c \in R$ & $\omega$ is the cube root of unity)
$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$ ; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$ ;
$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$ ;
$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

**Ex.35** Prove that $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$, where $\omega$ is an imaginary cube root of unity.

**Sol.** We have $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Now $a^2 + b^2 + c^2 - ab - bc - ca$

$= a^2 + b^2 + c^2 + (\omega + \omega^2)ab + (\omega + \omega^2)bc + (\omega + \omega^2)ca$ $\quad [\because \omega + \omega^2 = -1]$

$= (a^2 + ab\omega + ac\omega^2) + (ba\omega^2 + b^2\omega^3 + bc\omega^4) + (ca\omega + cb\omega^2 + c^2\omega^3)$

$= a(a + b\omega + c\omega^2) + b\omega^2(a + b\omega + c\omega^2) + c\omega(a + b\omega + c\omega^2)$

$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$\therefore$ $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

**Ex.36** Let 'A' denotes the real part of the complex number $z = \dfrac{19+7i}{9-i} + \dfrac{20+5i}{7+6i}$

and 'B' denotes the sum of the imaginary parts of the roots of the equation $z^2 - 8(1 - i)z + 63 - 16i = 0$

and 'C' denotes the sum of the series, $1 + i + i^2 + i^3 + ..... + i^{2008}$ where $i = \sqrt{-1}$.

and 'D' denotes the value of the product $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$ where $\omega$ is the imaginary cube root of unity. Find the value of $\dfrac{A-B}{C+D}$.

**Sol.** A = Re (z)

now $z = \dfrac{(19+7i)(9+i)}{82} + \dfrac{(20+5i)(7-6i)}{85} = \dfrac{171+82i-7}{82} + \dfrac{140-120i+35i+30}{85}$

$= \dfrac{164+82i}{82} + \dfrac{170-85i}{85} = 2 + i + 2 - i \Rightarrow \quad z = 4 + 0i \Rightarrow \quad A = 4$

Let $\alpha = x + iy$, $\beta = a + ib \Rightarrow \alpha + \beta = (x + a) + i(y + b) = 8 - 8i$ $\quad \therefore \quad y + b = -8$

$\therefore$ sum of the imaginary parts of the roots of the equation $= -8$ $\quad \therefore \quad B = -8$

$S = 1 + i + i^2 + i^3 + ........ + i^{2008} = \dfrac{(1-i^{2009})}{1-i} = \dfrac{1-i}{1-i} = 1 \Rightarrow C = 1, D = 1$. Hence $\dfrac{A-B}{C+D} = \dfrac{4+8}{1+1} = 6$

**Ex.37** Let $Z = 18 + 26i$ where $Z_0 = x_0 + iy_0$ ($x_0, y_0 \in R$) is the cube root of Z having least positive argument. Find the value of $x_0y_0(x_0 + y_0)$.

**Sol.** Let $Z = 18 + 26i$. Let $r\cos\theta = 18$ and $r\sin\theta = 26$

$\therefore$ $r^2 = 324 + 676 = 1000 \Rightarrow r = 10\sqrt{10}$; $\tan\theta = \dfrac{26}{18} = \dfrac{13}{9}$; hence $\theta \in \left(0, \dfrac{\pi}{2}\right)$; $\dfrac{\theta}{3} \in \left(0, \dfrac{\pi}{6}\right)$

$\therefore$ $Z^{1/3} = [10\sqrt{10}(\cos\theta + i\sin\theta)]^{1/3} = 10\left[\cos\dfrac{\theta}{3} + i\sin\dfrac{\theta}{3}\right]$

now $\tan\theta = \dfrac{3\tan(\theta/3) - \tan^3(\theta/3)}{1 - 3\tan^2(\theta/3)} = \dfrac{3t - t^3}{1 - 3t^2}$ where $t = \tan\dfrac{\theta}{3}$

$\Rightarrow\ 13(1 - 3t^2) = 9(3t - t^3) \quad \Rightarrow \quad 13 - 39t^2 = 27t - 9t^3 \Rightarrow \quad 9t^3 - 39t^2 - 27t + 13 = 0$

$\Rightarrow\ 3t^2(3t - 1) - 12t(3t - 1) - 13(3t - 1) = 0 \quad \Rightarrow \quad (3t - 1)(3t^2 - 12t - 13) = 0$

$\therefore\ \tan\frac{\theta}{3} = \frac{1}{3} \Rightarrow \quad \frac{\theta}{3} \in \left(0, \frac{\pi}{6}\right) \quad \Rightarrow \quad \sin\frac{\theta}{3} = \frac{1}{\sqrt{10}}$ and $\cos\frac{\theta}{3} = \frac{3}{\sqrt{10}}$

$\therefore\ Z^{1/3} = 1 + 3i \quad \Rightarrow \quad x_0 = 1$ and $y_0 = 3 \quad \Rightarrow \quad x_0y_0\,(x_0 + y_0) = 12$

**Ex.38** If $(1 + x + x^2)^{3n} = \sum_{r=0}^{6n} a_r x^r$, then compute the value of , $a_0 + a_6 + a_{12} + \ldots\ldots + a_{6n}$ .

**Sol.** $(1 + x + x^2)^{3n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \ldots\ldots + a_{6n} x^{6n}$

Putting $x = 1 \Rightarrow 3^{3n} = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \ldots\ldots + a_{6n}$

$x = \omega \Rightarrow 0 = a_0 + a_1\omega + a_2\omega^2 + a_3 + a_4\omega + a_5\omega^2 + a_6 + \ldots\ldots + a_{6n}$

$x = \omega^2 \Rightarrow 0 = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 + a_5\omega + a_6 + \ldots\ldots + a_{6n}$

$x = -1 \Rightarrow 1 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - \ldots\ldots + a_{6n}$

$x = -\omega \Rightarrow (-2)^{3n} = a_0 - a_1\omega + a_2\omega^2 - a_3 + a_4\omega - a_5\omega^2 + a_6 - \ldots\ldots + a^{6n}$

$x = -\omega^2 \Rightarrow (-2)^{3n} = a_0 - a_1\omega^2 + a_2\omega - a_3 + a_4\omega^2 - a_5\omega + a_6 - \ldots\ldots + a_{6n}$

Adding $\frac{3^{3n} + 1 + (-1)^{3n}\ .\ 2^{3n+1}}{6} = a_0 + a_6 + \ldots\ldots + a_{6n}$

**Ex.39** If ' n ' is odd and not a multiple of 3 , prove that $x\,(x + 1)\,(x^2 + x + 1)$ is a factor of $(x + 1)^n - x^n - 1$ , $n \in N$ .

**Sol.** Let $f(x) = (x + 1)^n - x^n - 1$

put $x = 0$ ; $f(0) = 0 \Rightarrow$ x is factor of f (x)

put $x = -1$ ; $f(-1) = -(-1)^n - 1 = 1 - 1 = 0 \Rightarrow$ $x + 1$ is a factor of f (x) .

Put $x = \omega$ ; $f(\omega) = (1 + \omega)^n - \omega^n - 1 \quad = \quad \omega^{2p}\ .\ \omega^{2n} - \omega^n - 1 \quad = \omega^{2p} \quad = -\omega^{2n} - (\omega^n + 1)$

$= -\omega^{2n} - [1 + \omega^n + \omega^{2n} - \omega^{2n}] \quad = -\omega^{2n} + \omega^{2n} \quad = 0\ (1 + \omega^n + \omega^{2n} = 0)$

Similarly put $x = \omega^2$ and proved $f(\omega^2) = 0$ . Hence the expression

$x\,(x + 1)\,(x^2 + x + 1) = x\,(x + 1)\,(x - \omega)\,(x - \omega^2)$ divides f (x). $\therefore z = -1 - i$

**Ex.40** If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots + C_n x^n$ $(n \in N$ ; $C_r = {}^nC_r)$ and

$S_1 = C_0 + C_3 + C_6 + \ldots\ldots.$ ; $\quad S_2 = C_1 + C_4 + C_7 + \ldots\ldots.$; $\quad S_3 = C_2 + C_5 + C_8 + \ldots\ldots.$

Show that the values of $S_1$, $S_2$ & $S_3$ are respectively given by , $\frac{1}{3}\left(2^n + 2\cos\frac{r\pi}{3}\right)$

with , $r = n$ for $S_1$ ; $\quad r = n - 2$ for $S_2$ & $r = (n + 2)$ for $S_3$.

**Sol.** Putting $x = 1$ , $\omega$ & $\omega^2$ in the expansion of $(1 + x)^n$ where $\omega$ is the cube root of unity.

$2^n = C_0 + C_1 + C_2 + \ldots\ldots. + C_n$ ........(1)

$(1 + \omega)^n = C_0 + C_1\omega + C_2\omega^2 + \ldots\ldots. + C_n\omega^n$ ........(2)

$(1 + \omega^2)^n = C_0 + C_1\omega^2 + C_2\omega^4 + \ldots\ldots. + C_n\omega^{2n}$ ........(3)

Adding $3S_1 = 2^n + (1+\omega)^n + (1+\omega^2)^n = 2^n + 2\,\text{Re part of }(1+\omega)^n = 2^n + 2\cos\frac{n\pi}{3}$ (verify)

$\therefore \quad S_1 = \frac{1}{3}\left(2^n + 2\cos\frac{r\pi}{3}\right)$ where $r = n$

Again (1) + (2) x $\omega^2$ + (3) x $\omega$ gives

$3S_2 = 2^n + \omega^2(1+\omega)^n + \omega(1+\omega^2)^n = 2^n + 2\,\text{Re part of }\ \omega^2(1+\omega)^n = 2^n + 2\cos\frac{n-2}{3}\pi$

$\therefore \quad S_2 = \frac{1}{3}\left[2^n + 2\cos\frac{r\pi}{3}\right]$ where $r = n - 2$. Similarly $S_3$ can be found out

## H. N$^{th}$ ROOTS OF UNITY

If $1, \alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{n-1}$ are the n, n$^{th}$ root of unity then

**(i)** They are in G.P. with common ratio $e^{i(2\pi/n)}$

**(ii)** $1^p + \alpha_1^p + \alpha_2^p + \ldots. + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

$= n$ if p is an integral multiple of n

**(iii)** $(1-\alpha_1)(1-\alpha_2)\ldots\ldots(1-\alpha_{n-1}) = n$

$(1+\alpha_1)(1+\alpha_2)\ldots\ldots.(1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

**(iv)** $1 . \alpha_1 . \alpha_2 . \alpha_3 \ldots\ldots\ldots \alpha_{n-1} = 1$ or $-1$ according as n is odd or even.

**Ex.41** Find the 10$^{th}$ roots of unity and show that the product of any two of them is again one of the 10$^{th}$ roots.

**Sol.** For r = 0, 1, 2, ....., 9, the 10$^{th}$ roots of unity are given by $z^{10} = 1 = \cos(2r\pi) + i\sin(2r\pi)$

So by De Moivre's theorem, $z = [\cos(2r\pi) + i\sin(2r\pi)]^{1/10} = \cos\frac{2r\pi}{10} + i\sin\frac{2r\pi}{10} = \omega^r$,

where $\omega = \cos(\pi/5) + i\sin(\pi/5)$. Let $z_1 = \omega^r$ and $z_2 = \omega^s$ $(0 \le r, s \le 9)$ be two of these 10$^{th}$ roots.

Then $z_1z_2 = \omega^{r+s} = \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^{r+s} = \cos\left[(r+s)\frac{\pi}{5}\right] + i\sin\left[(r+s)\frac{\pi}{5}\right]$.

If $0 \le r + s \le 9$, then $z_1z_2$ is also a 10$^{th}$ root of unity. On the other hand, if $r + s \ge 10$, let $r + s = 10 + k$, where $0 \le k \le 8$, so that $z_1z_2 = \cos\left(2\pi + \frac{k\pi}{5}\right) + i\sin\left(2\pi + \frac{k\pi}{5}\right) = \cos\frac{k\pi}{5} + i\sin\frac{k\pi}{5}$,

showing that $z_1z_2$ is a 10$^{th}$ root of unit in general.

**Ex.42** Determine the value of z when $z^6 = \sqrt{3} + i$

**Sol.** $\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left\{\cos\left(2k\pi + \frac{\pi}{6}\right) + i\sin\left(2k\pi + \frac{\pi}{6}\right)\right\}$, k = any integer.

Now $z = \left(\sqrt{3} + i\right)^{1/6} = 2^{1/6}\left\{\cos\frac{2k\pi + \frac{\pi}{6}}{6} + i\sin\frac{2k\pi + \frac{\pi}{6}}{6}\right\}$, where k = 0, 1, 2, 3, 4, 5.

**Ex.43** Find the real factors of $x^6 + 1$.

**Sol.** To factorise $x^6 + 1$, we first find the roots of $x^6 + 1 = 0$.

$x^6 + 1 = 0$ or $x^6 = -1 = \cos(2k + 1)\pi + i \sin(2k + 1)\pi$,

or $x = \cos\frac{2k+1}{6}\pi + i \sin\frac{2k+1}{6}\pi$, where $k = 0, 1, 2, 3, 4, 5$.

$$\therefore \quad x = \begin{cases} \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}, \quad \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}, \quad \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} \\ \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}, \quad \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}, \quad \cos\frac{\pi}{6} - i\sin\frac{\pi}{6} \end{cases}$$

or $x = \frac{\sqrt{3}}{2} + i\frac{1}{2}, \frac{\sqrt{3}}{2} - i\frac{1}{2}, i, -i, -\frac{\sqrt{3}}{2} + i\frac{1}{2}, -\frac{\sqrt{3}}{2} - i\frac{1}{2}$.

Hence $x^6 - 1 = (x - i)(x + i)\left(x - \frac{\sqrt{3}}{2} - \frac{i}{2}\right)\left(x - \frac{\sqrt{3}}{2} + \frac{i}{2}\right) \times \left(x + \frac{\sqrt{3}}{2} - \frac{i}{2}\right)\left(x + \frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= (x^2 + 1)\left\{\left(x - \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}\right\}\left\{\left(x + \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}\right\} = (x^2 + 1)(x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)$.

**Ex.44** Resolve $z^7 - 1$ into linear and quadratic factors and hence deduce that $\cos\frac{\pi}{7} . \cos\frac{2\pi}{7} . \cos\frac{4\pi}{7} = \frac{1}{8}$.

**Sol.** $7^{th}$ roots are $1, \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right), \left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right)$, .......

Hence $(z^7 - 1) = (z - 1)\left(z^2 - 2\cos\frac{2\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{6\pi}{7}z + 1\right)$

Put $z = i$

$-(i + 1) = +(i - 1)\left(-2\cos\frac{2\pi}{7}i\right)\left(-2\cos\frac{4\pi}{7}i\right)\left(-2\cos\frac{6\pi}{7}i\right)$

$\Rightarrow -(i + 1) = (-1 - i)\, 8\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} . \cos\frac{6\pi}{7} \quad \Rightarrow \quad 1 = 8\cos\frac{\pi}{7} . \cos\frac{2\pi}{7} . \cos\frac{4\pi}{7}$

**Ex.45** If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z - 2)(z^2 - pz + 4)(z^2 - qz + 4)$ then find the value of $(p^2 + 2p)$.

**Sol.** $z^5 - 32 = (z - z_0)(z - z_1)(z - z_2)(z - z_3)(z - z_4) = \prod_{i=0}^{4}(z - z_i)$

where $z_{i's}$ , i = 0, 1, 2, 3, 4 are given by $z_i = 2\left(\cos\frac{2m\pi}{5} + i\sin\frac{2m\pi}{5}\right)$ (using Demoivre's Theorem)

with m = 0, 1, 2, 3, 4, we get $z_0 = 2$, and $z_1 = 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$; $z_2 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$;

$z_3 = 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$; $z_4 = 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$

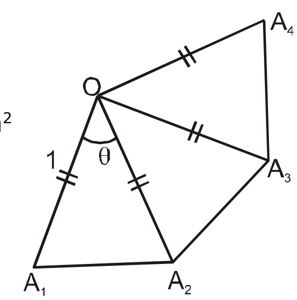
hence, $z^5 - 32 = (z - 2)\left(z^2 - 4\cos\frac{2\pi}{5}z + 4\right)\left(z^2 - 4\cos\frac{4\pi}{5}z + 4\right)$ $\left[\begin{array}{l}\text{using } (z-\alpha)(z-\overline{\alpha}) \\ \quad = z^2 - (\alpha + \overline{\alpha})z + \alpha\overline{z}\end{array}\right]$

$\therefore$ $p = 4\cos\frac{2\pi}{5} = 4\cos 72° = 4\sin 18°$

$\therefore$ $p^2 + 2p = [16\sin^2 18° + 8\sin 18°] = 8[1 - \cos 36° + \sin 18°] = 8\left[1 + \frac{\sqrt{5}-1}{4} - \frac{\sqrt{5}+1}{4}\right] = 4$

**Ex.46** Let $A_k$ (k = 1, 2, .... n) be the vertices of a regular m – polygon inscribed in a unit circle then prove that, $\prod_{r=2}^{r=n}|A_1 A_r| = n$ .

**Sol.** If $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are $n^{th}$ roots of unity then,




$(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{n-1}) = n \Rightarrow |1 - \alpha_1|^2\,|1 - \alpha_2|^2 \ldots\ldots |1 - \alpha_{n-1}|^2 = n^2$

when $\alpha_1 = \cos\theta + \sin\theta$ and $\theta = 2\pi/n$

$(1 - \alpha_1)(1 - \overline{\alpha}_1)\ldots\ldots(1 - \alpha_{n-1})(1 - \overline{\alpha}_{n-1}) = n^2$

Now $(1 - \alpha_1)(1 - \overline{\alpha}_1) = 1 - (\alpha_1 + \overline{\alpha}_1) + \alpha_1\overline{\alpha}_1 = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2}$

$\Rightarrow 2\sin\frac{\theta}{2} \cdot 2\sin\frac{2\theta}{2} \cdot 2\sin\frac{3\theta}{2} \ldots\ldots 2\sin\frac{(n+1)\theta}{2} = n \Rightarrow |A_1A_2|\,|A_1A_3|\,|A_1A_4| \ldots\ldots |A_1A_n| = n$

**Ex.47** Find the roots of $z^n = (z + 1)^n$ and show that the points which represent them are collinear. Hence show that these roots are also the roots of the equation,

$$\left(2\sin\frac{m\pi}{n}\right)^2 \bar{z}^2 + \left(2\sin\frac{m\pi}{n}\right)^2 \bar{z} + 1 = 0$$ where $m = 1, 2, 3, \ldots, (n-1)$ & $|z|$ is finite.

**Sol.** $\frac{z+1}{z} = (1)^{1/n} \Rightarrow z = -\frac{1}{2}\left[1 + i\cot\frac{m\pi}{n}\right], \bar{z} = -\frac{1}{2}\left[1 - i\cot\frac{m\pi}{n}\right]$

or $2\bar{z} = \frac{i\cos\frac{m\pi}{n} - \sin\frac{m\pi}{n}}{\sin\frac{m\pi}{n}}$ $m \ne 0$, $|z|$ = finite

$2\bar{z}\sin\frac{m\pi}{n} + \sin\frac{m\pi}{n} = i\cos\frac{m\pi}{n}$ ; square and get the result

## I. SUM OF IMPORTANT SERIES

**(i)** $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{n+1}{2}\right)\theta$.

**(ii)** $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin\left(\frac{n+1}{2}\right)\theta$.

**Remark :** If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

**Ex.48** If $\theta \ne k\pi$, show that $\cos\theta\sin\theta + \cos^2\theta\sin 2\theta + \ldots + \cos^n\theta\sin n\theta = \cot\theta(1 - \cos^n\theta\cos n\theta)$

**Sol.** $S = \cos\theta\sin\theta + \cos^2\theta\sin 2\theta + \ldots + \cos^n\theta\sin n\theta$

and $C = \cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \ldots + \cos^2\theta\cos n\theta$

so that $C + iS = \cos\theta\, z + \cos^2\theta\, z^2 + \ldots + \cos^n\theta\, z^n$, where $z = \cos\theta + i\sin\theta$. In other words,

$$C + iS = \frac{\cos\theta z[1 - (\cos\theta z)^n]}{1 - \cos\theta z} = \frac{\cos\theta(\cos\theta + i\sin\theta)\,[1 - \cos^n\theta(\cos n\theta + i\sin n\theta)]}{1 - \cos\theta(\cos\theta + i\sin\theta)} \quad [\theta \ne k\pi]$$

$$= \frac{\cos\theta(\cos\theta + i\sin\theta)\,[1 - \cos^n\theta(\cos n\theta + i\sin n\theta)]}{\sin^2\theta - i\cos\theta\sin\theta} = \frac{\cos\theta(\cos\theta + i\sin\theta)\,[1 - \cos^n\theta(\cos n\theta + i\sin n\theta)]}{-i\sin\theta(\cos\theta + i\sin\theta)}$$

$= i\cot\theta(1 - \cos^n\theta\cos n\theta - i\cos^n\theta\sin n\theta)$

Equating the imaginary parts we therefore get $S = \cot\theta(1 - \cos^n\theta\cos n\theta)$.

**Ex.49** Find the sum of the infinite series, $\sin\alpha + \frac{1}{2}\sin 2\alpha + \frac{1}{2^2}\sin 3\alpha + \frac{1}{2^3}\sin 4\alpha + \ldots\ldots \infty$.

**Sol.** Let $S = \sin\alpha + \frac{1}{2}\sin 2\alpha + \ldots\ldots$ and $C = \cos\alpha + \frac{1}{2}\cos 2\alpha + \ldots\ldots$ $r = \left|\frac{1}{2}e^{i\alpha}\right| = \frac{1}{2} < 1$

$= \frac{e^{i\alpha}}{1 - \frac{1}{2}e^{i\alpha}} = \frac{2e^{i\alpha}}{2 - e^{i\alpha}}$, simplifying $C + iS = \frac{4\cos\alpha - 2}{5 - 4\cos\alpha} + i\,\frac{4\sin\alpha}{5 - 4\cos\alpha}$

**Ex.50** Use complex numbers to prove that the sum, $\sum_{r=0}^{n-1} \cos^2\left(\alpha + \frac{\alpha\pi}{n}\right) = \frac{n}{2}$ where $n \in N$, $n \geq 2$.

**Sol.** L H S $= \cos^2\alpha + \cos^2(\alpha + \theta) + \cos^2(\alpha + 2\theta) + \ldots\ldots + \cos^2(\alpha + (n-1)\theta)$ where $\theta = \frac{\pi}{n}$

$$= \frac{n}{2} + \frac{1}{2}\left[\cos 2\alpha + \cos 2(\alpha+\theta) + \cos 2(\alpha + 2\theta) + \ldots\ldots + \cos 2\left(\alpha + \overline{n-1}\,\theta\right)\right]$$

Consider $c = \cos 2\alpha + \cos 2(\alpha+\theta) + \ldots\ldots + \cos 2\left(\alpha + \overline{n-1}\,\theta\right)$ and

$s = \sin 2\alpha + \sin 2(\alpha+\theta) + \ldots\ldots + \sin 2\left(\alpha + \overline{n-1}\,\theta\right)$

$$\Rightarrow \quad c + is = e^{i2\alpha}\left[1 + e^{i2\theta} + e^{i4\theta} + \ldots\ldots + e^{i2\left(\overline{n-1}\,\theta\right)}\right]$$

$$= e^{i2\alpha}\frac{\left[e^{i2n\theta}-1\right]}{e^{2i\theta}-1} = \frac{e^{i2\alpha}\,.\,e^{in\theta}\left[e^{in\theta}-e^{-in\theta}\right]}{e^{i\theta}\left[e^{i\theta}-e^{-i\theta}\right]} = e^{i\left(2\alpha + \overline{n-1}\,\theta\right)}\frac{\left[2i\sin n\theta\right]}{2i\sin\theta}$$

Equating real part $\quad c = \frac{\sin n\theta}{\sin\theta}\left[\cos\left(2\alpha + \overline{n-1}\,\theta\right)\right] = 0 \quad$ if $\theta = \frac{\pi}{n}$

**Ex.51** If A and B are supplementary angles and n is odd integer, then prove that $\sum_{r=0}^{n} {}^nC_r \cos(nA + r(B-A)) = 0$

**Sol.** $A + B = \pi \quad \Rightarrow \quad e^{iA} + e^{iB} = 2i\sin A \quad \Rightarrow \quad (e^{iA} + e^{iB}) = (2i \sin A)^n$

As n is odd integer. $\Rightarrow$ Real part of $(e^{iA} + e^{iB})^n = 0$ $\Rightarrow$ Real part of $\sum_{r=0}^{n} {}^nC_r\, e^{i(n-r)A}\, e^{iBr} = 0$

$$\Rightarrow \quad \sum_{r=0}^{n} {}^nC_r \cos(nA + r(B - A)) = 0.$$

## J. STRAIGHT LINES & CIRCLES IN COMPLEX NUMBERS

**(1)** If $z_1$ & $z_2$ are two complex numbers then the complex number $z = \frac{nz_1 + mz_2}{m+n}$ divides the joins of $z_1$ & $z_2$ in the ratio m : n.

**Remark :**

**(i)** If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers $z_1$, $z_2$ & $z_3$ are collinear.

**(ii)** If the vertices A, B, C of a $\Delta$ represent the complex nos. $z_1$, $z_2$, $z_3$ respectively, then

**(a)** Centroid of the $\Delta$ ABC $= \dfrac{z_1 + z_2 + z_3}{3}$

**(b)** Orthocentre of the $\Delta$ ABC $= \dfrac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C}$ **OR** $\dfrac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$

**(c)** Incentre of the $\Delta$ ABC $= (az_1 + bz_2 + cz_3) \div (a + b + c)$.

**(d)** Circumcentre of the $\Delta$ ABC $= (Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

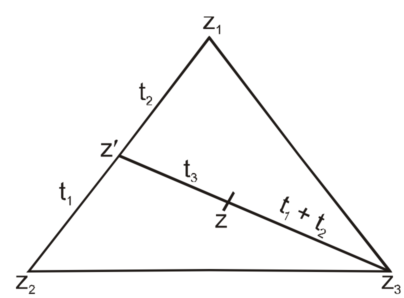
**Ex.52** Prove that the roots of the equation $\dfrac{1}{z - z_1} + \dfrac{1}{z - z_2} + \dfrac{1}{z - z_3} = 0$ where $z_1, z_2, z_3$ are pairwise distinct complex numbers, correspond to points on a complex plane which lie inside a triangle with vertices $z_1, z_2, z_3$ or on its sides.

**Sol.** $\dfrac{\bar{z} - \bar{z}_1}{|z - z_1|^2} + \dfrac{\bar{z} - \bar{z}_2}{|z - z_2|^2} + \dfrac{\bar{z} - \bar{z}_3}{|z - z_3|^2} = 0 \Rightarrow \dfrac{\bar{z} - \bar{z}_1}{a} + \dfrac{\bar{z} - \bar{z}_2}{b} + \dfrac{\bar{z} - \bar{z}_3}{c} = 0$ or $\dfrac{z - z_1}{a} + \dfrac{z - z_2}{b} + \dfrac{z - z_3}{c} = 0$

where $|z - z_1|^2 = a$ etc $|z - z_2|^2 = b$, $|z - z_3|^2 = c$ $\quad\Rightarrow\quad bc(z - z_1) + zc(z - z_2) + ab(z - z_3) = 0$

Let $bc = t_1$ ; $ca = t_2$ ; $ab = t_3$ $\quad\Rightarrow\quad t_1(z - z_1) + t_2(z - z_2) + t_3(z - z_3) = 0$

$\Rightarrow\ (t_1 + t_2 + t_3)z = t_1z_1 + t_2z_2 + t_3z_3 \Rightarrow\ z = \dfrac{t_1z_1 + t_2z_2 + t_3z_3}{t_1 + t_2 + t_3}$




$= \dfrac{t_1z_1 + t_2z_2}{t_1 + t_2} \cdot \dfrac{t_1 + t_2}{t_1 + t_2 + t_3} + \dfrac{t_3z_3}{t_1 + t_2 + t_3} = \dfrac{t_1 + t_2}{t_1 + t_2 + t_3} z' + \dfrac{t_3z_3}{t_1 + t_2 + t_3}$

$\Rightarrow\ z = \dfrac{(t_1 + t_2)z' + t_3z_3}{t_1 + t_2 + t_3} \quad\Rightarrow\quad$ z lies inside the $\Delta\, z_1 z_2 z_3$

If $t_1 = t_2 = t_3 \Rightarrow z$ is the centroid of the triangle . Also if $a = b = c$

$\Rightarrow\ |z - z_1| = |z - z_2| = |z - z_3| \Rightarrow z$ is the circumcentre. if $t_3 = 0 \Rightarrow z$ lies on the line joining $z_1$ and $z_2$

**Ex.53** $z_1, z_2$ and $z_3$ are the vertices of a triangle ABC such that $|z_1| = |z_2| = |z_3|$ and AB = AC. Prove that $\dfrac{(z_1 + z_2)(z_1 + z_2)}{(z_2 + z_3)^2}$ is purely real.

**Sol.** Since $|z_1| = |z_2| = |z_3| \Rightarrow$ Circumcentre of $\Delta$ABC is at the origin $\Rightarrow$ Orthocentre $(z_p)$ of $\Delta$ABC is $z_1 + z_2 + z_2$

Also angle subtended by AB and AC at the orthocentre are A + B and A + C respectively.

$\Rightarrow$ angles subtended by the sides AB and AC at the orthocentre are equal.

$\Rightarrow\ \arg\left(\dfrac{z_1 + z_2 + z_3 - z_1}{z_1 + z_2 + z_3 - z_3}\right) = \arg\left(\dfrac{z_1 + z_2 + z_3 - z_2}{z_1 + z_2 + z_3 - z_1}\right) \quad\Rightarrow\quad \arg\left(\dfrac{z_2 + z_3}{z_1 + z_2}\right) = \arg\left(\dfrac{z_1 + z_3}{z_2 + z_3}\right)$

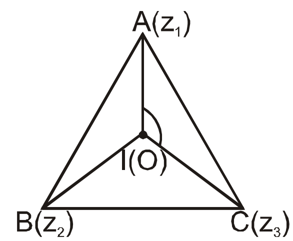
$\Rightarrow\ \arg\left(\dfrac{(z_1 + z_3)(z_1 + z_2)}{(z_2 + z_3)^2}\right) = 0 \Rightarrow \dfrac{(z_1 + z_3)(z_1 + z_2)}{(z_2 + z_3)^2}$ is purely real.

**Ex.54** $z_1$, $z_2$ and $z_3$ are the vertices of an isosceles triangle in anticlockwise direction with origin as incentre. If $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) > \frac{\pi}{2}$, then prove that $z_2$, $z_1$ and $kz_3$ are in G.P. where $k \in R^+$.

**Sol.** Let $z_1$, $z_2$ and $z_3$ represent the vertices A, B and C respectively of triangle ABC.
Now $\angle A > 90°$ and triangle is isosceles $\Rightarrow$ $\angle B = \angle C$.

$\angle AIC = \frac{\pi}{2} + \frac{B}{2}$ and $\angle BIA = \frac{\pi}{2} + \frac{C}{2}$ $\Rightarrow$ $\frac{z_1}{z_3} = \left|\frac{z_1}{z_3}\right| e^{i\left(\frac{\pi}{2}+\frac{B}{2}\right)}$ .....(1) and $\frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{i\left(\frac{\pi}{2}+\frac{C}{2}\right)}$ ....(2)

on dividing equation (1) by equation (2) we get $\frac{z_1^2}{z_2 z_3} = \frac{|z_1|^2}{|z_2||z_3|} e^{i\left(\frac{B-C}{2}\right)}$

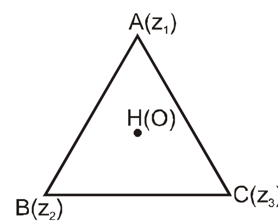



$\Rightarrow$ $\frac{z_1^2}{z_2 z_3} = \frac{|z_1|^2}{|z_2||z_3|}$ = positive real number ($\because$ $\angle B = \angle C$)

$\Rightarrow$ $z_1^2 = kz_2z_3$ where $k \in R^+$. Hence $z_2$, $z_1$ and $kz_3$ are in G.P.

**Ex.55** Let A($z_1$), B($z_2$) and C($z_3$) forms an acute angled triangle in argand plane having origin as it's orthocentre. Prove that $z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1$.

**Sol.** Clearly the angle between BC and AH is $\frac{\pi}{2}$




$\Rightarrow$ $\frac{z_3 - z_2}{z_1}$ is purely imaginary, $\Rightarrow$ $\frac{z_3 - z_2}{z_1} + \frac{\bar{z}_3 - \bar{z}_2}{\bar{z}_1} = 0$

$\Rightarrow$ $z_3\bar{z}_1 - z_2\bar{z}_1 + z_1\bar{z}_3 - z_2\bar{z}_1 = 0$ $\Rightarrow$ $z_3\bar{z}_1 + z_1\bar{z}_3 = z_1\bar{z}_2 - z_2\bar{z}_1$

We also have $\frac{z_3 - z_1}{z_2} + \frac{\bar{z}_3 - \bar{z}_1}{\bar{z}_2} = 0$ $\Rightarrow$ $\bar{z}_2 z_3 + z_2\bar{z}_3 = \bar{z}_1 z_2 + z_1\bar{z}_2$ $\Rightarrow$ $z_1\bar{z}_2 + z_1\bar{z}_1 = \bar{z}_2 z_3 + z_3\bar{z}_2 = z_3\bar{z}_1 + z_1\bar{z}_3$

**(b)** amp(z) = $\theta$ is a ray emanating from the origin inclined at an angle $\theta$ to the x– axis .

**(c)** $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b .

**(d)** The equation of a line joining $z_1$ & $z_2$ is given by ; $z = z_1 + t\,(z_1 - z_2)$ where t is a parameter.

**(e)** $z = z_1\,(1 + it)$ where t is a real parameter is a line through the point $z_1$ & perpendicular to $oz_1$.

**(f)** The equation of a line passing through $z_1$ & $z_2$ can be expressed in the determinant form as

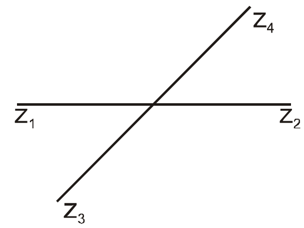
$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$ . This is also the condition for three complex numbers to be collinear.

**(g)** Complex equation of a straight line through two given points $z_1$ & $z_2$ can be written as $z\left(\bar{z}_1 - \bar{z}_2\right) - \bar{z}\left(z_1 - z_2\right) + \left(z_1\bar{z}_2 - \bar{z}_1 z_2\right) = 0$, which on manipulating takes the form as $\bar{\alpha} z + \alpha\bar{z} + r = 0$ where r is real and $\alpha$ is a non zero complex constant.

**Ex.56** A and B represent $z_1$ and $z_2$ in the Argands plane . The complex slope of AB is defined to be $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$ .

Prove that the two lines in the Argand's plane with complex slopes $\omega_1$ and $\omega_2$ will be perpendicular if and only if $\omega_1 + \omega_2 = 0$ . Also find the condition for two lines with complex slopes $\omega_1$ and $\omega_2$ to be parallel.

**Sol.** If $l_1$ is perpendicular to $l_2$ then $\dfrac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary




$\Rightarrow \dfrac{z_1 - z_2}{z_3 - z_4} + \dfrac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4} = 0 \;\Rightarrow\; \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} + \dfrac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4} = 0$

$\Rightarrow \omega_1 + \omega_2 = 0$ , similarly for parallel $\omega_1 - \omega_2 = 0$

**(h)** The equation of circle having centre $z_0$ & radius $\rho$ is

$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form

$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ , r is real centre $-\alpha$ & radius $\sqrt{\alpha\bar{\alpha} - r}$ . Circle will be real if $\alpha\bar{\alpha} - r \geq 0$.

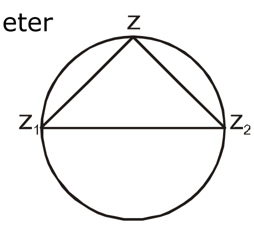
**(i)** The equation of the circle described on the line segment joining $z_1$ & $z_2$ as diameter is

arg $\dfrac{z - z_2}{z - z_1} = \pm \dfrac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

**(j)** Condition for four given points $z_1$, $z_2$, $z_3$ & $z_4$ to be concyclic is , the number $\dfrac{z_3 - z_1}{z_3 - z_2} \cdot \dfrac{z_4 - z_2}{z_4 - z_1}$ is real. Hence the equation of a circle through 3 non collinear points $z_1$, $z_2$ & $z_3$ can be taken as

$\dfrac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ is real $\Rightarrow \dfrac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} = \dfrac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$

**Ex.57** Show that the equation of the circle in the complex plane with $z_1$ & $z_2$ as its diameter can be expressed as , $2z\bar{z} - (\bar{z}_1 + \bar{z}_2)z - (z_1 + z_2)\bar{z} + z_1\bar{z}_2 + z_2\bar{z}_1 = 0$ .




**Sol.** arg $\dfrac{z - z_1}{z - z_2} = \pm \dfrac{\pi}{2} \Rightarrow \dfrac{z - z_1}{z - z_2}$ is purely imaginary $\Rightarrow \dfrac{z - z_1}{z - z_2} + \dfrac{\bar{z} - \bar{z}_1}{\bar{z} - \bar{z}_2} = 0 \Rightarrow$ Result

**Ex.58** Prove that the equations , $\left|\dfrac{z - 1}{z + 1}\right|$ = constant and amp$\left(\dfrac{z - 1}{z + 1}\right)$ = constant on the argand plane represent the equations of the circles which are orthogonal.

**Sol.** Put $\left|\dfrac{z - 1}{z + 1}\right| = \lambda$. This simplifies to $x^2 + y^2 + 2\dfrac{\lambda^2 + 1}{\lambda^2 - 1}x + 1 = 0$ or $x^2 + y^2 + 2gx + 1 = 0$

Similarly the other equation , amp$\left(\dfrac{z - 1}{z + 1}\right) = \mu$ reduces to, $x^2 + y^2 - \dfrac{2}{\mu}y + 1 = 0$ or $x^2 + y^2 + 2fy - 1 = 0$

**Ex.59** In the equation, $z^2 + 2\lambda z + 1 = 0$, $\lambda$ is a parameter which can take any real value. Show that if $-1 < \lambda < 1$, the roots of this equation lie on a certain circle in the Argand diagram, but that if $\lambda > 1$, one root lies inside the unit circle and one outside. Prove that for very large values of $\lambda$, the roots are approximately $-2\lambda$, $-1/2\lambda$.

**Sol.** $z^2 + 2\lambda z + 1 = 0 \Rightarrow z = \dfrac{-2\lambda \pm \sqrt{4\lambda^2 - 4}}{2}$ or $z = -\lambda \pm \sqrt{\lambda^2 - 1}$

or $z = -\lambda \pm \sqrt{1-\lambda^2}\ i$ (if $-1 < \lambda < 1$) or $z = -\lambda \pm \mu i$ where $\sqrt{1-\lambda^2} = \mu > 0$

$\therefore z + \lambda = \mu i$ or $z + \lambda = -\mu i \Rightarrow \overline{z+\lambda} = -\mu i$ or $\overline{z+\lambda} = \mu i$

Hence $(z+\lambda)\left(\overline{z+\lambda}\right) = \mu^2$ or $(z+\lambda)\left(\overline{z+\lambda}\right) = \mu^2 \Rightarrow |z + \lambda| = \mu$

Hence 'z' lies on a circle with centre $-\lambda$ and radius $\mu$.

Again let $\lambda > 1$, we have $z = -\lambda + \sqrt{\lambda^2 - 1}$ or $-\lambda - \sqrt{\lambda^2 - 1}$

Hence $z = -\lambda + \mu$ or $z = -\lambda - \mu$ where $\mu^2 = \lambda^2 + 1$ ; $z = -\lambda \pm \mu$.

If $z_1$ & $z_2$ are the roots then, $|z_1| = |-\lambda + \mu|$ & $|z_2| = |\lambda + \mu|$ ; $|z_1||z_2| = |\mu^2 - \lambda^2| = 1$

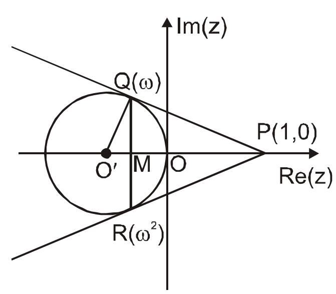
$\Rightarrow$ if $|z_1| < 1$ then $|z_2| > 1$ i.e. one root lies inside the unit circle and the other outside the unit circle. Further, if $\lambda$ is large, $\lambda^2 = \mu^2 + 1 \Rightarrow \lambda \cong \mu$

Hence the roots are $z_1 = -\lambda - \mu \cong -2\lambda$ and $z_2 = -\lambda + \mu = \dfrac{(-\lambda+\mu)(-\lambda-\mu)}{(-\lambda-\mu)} = \dfrac{\lambda^2 - \mu^2}{-\lambda-\mu} = -\dfrac{1}{2\lambda}$

**Ex.60** Find the equation of the circle in argand plane which passes through non real cube roots of unity and touches two sides of triangle with vertices as cube roots of unity.

**Sol.** Clearly triangle PQR is an equilateral triangle. Now, $QM = \dfrac{1}{2}|\omega - \omega^2| = \dfrac{1}{2}\left|2\dfrac{\sqrt{3}}{2}\right| = \dfrac{\sqrt{3}}{2}$ units.

In $\Delta QO'M$, $QO' = \dfrac{QM}{\sin 60^\circ} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = 1$ unit




And $O'O = QO' = 1$ unit (radius of circle)

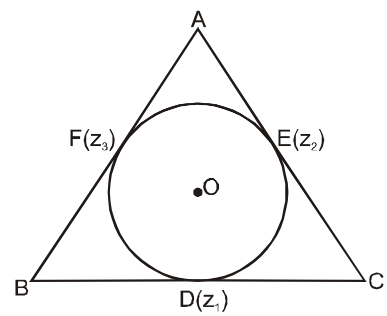
Point $O'$ is given by $(-1, 0)$

$\Rightarrow$ equation of circle $|z + 1| = 1 \Rightarrow z\bar{z} + z + \bar{z} = 0$

**Ex.61** Complex numbers $z_1$, $z_2$, $z_3$ are represented by the points of contact D, E, F of the incircle of triangle ABC, with the centre O of the incircle taken as the origin. If BO meets DE at G, find the complex number represented by G.

**Sol.** Let te incircle touch AB at F. Let O be the origin and let $z_1$, $z_2$, $z_3$ be the complex numbers represented by D, E and F respectively. Since C is the point of intersection of the tangents to the circle at D and E, C represents the complex number $\frac{2z_1z_2}{z_1+z_2}$. Similarly A and B represent the complex numbers $\frac{2z_2z_3}{z_2+z_3}$ and $\frac{2z_1z_3}{z_1+z_3}$. Let r be the radius of the incircle. $\Rightarrow |z_1| = |z_2| = |z_3| = r$.

Equation of the line BO is $\frac{z}{\frac{2z_1z_3}{z_1+z_3}} = \frac{\bar{z}}{\frac{2\bar{z}_1\bar{z}_3}{\bar{z}_1+\bar{z}_3}} = \frac{\bar{z}}{\frac{2z_1\bar{z}_1z_3\bar{z}_3}{r^2(\bar{z}_1+\bar{z}_3)}}$ or $z = \sqrt{\frac{z_1\bar{z}_1z_3\bar{z}_3}{\bar{z}_1\bar{z}_3}}\,\bar{z}$



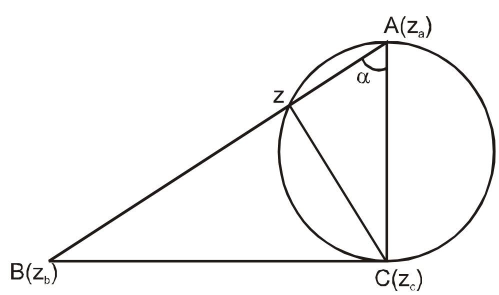

$\Rightarrow \sqrt{\bar{z}_1\bar{z}_3}\,z = \sqrt{z_1z_3}\,\bar{z}$ ,,,(1)

Equation of line DE is $\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$. Where it meets (1), we have

$$\frac{z-z_1}{z_2-z_1} = \frac{z.\sqrt{\frac{\bar{z}_1\bar{z}_3}{z_1z_3}}-\bar{z}_1}{\bar{z}_2-\bar{z}_1} = \frac{\frac{zr^2}{z_1z_3}-\frac{r^2}{z_1}}{\frac{r^2}{z_2}-\frac{r^2}{z_1}} \quad \Rightarrow \quad z = \frac{(z_1-z_2)z_3}{z_2+z_3}$$ which is represented by G.

**Ex.62** $A(z_a)$, $B(z_b)$, $C(z_c)$ are vertices of right angled triangle, $z_c$ being the orthocentre. A circle is described on AC as diameter. Find the point of intersection of the circle with hypotenuse.

**Sol.** Let d be the point o intersection




In $\Delta$ ADC $\frac{z_a-z}{|z_a-z|} = \frac{z_c-z}{|z_c-z|}e^{i\pi/2}$

In $\Delta$ ABC $\frac{z_b-z_c}{|z_b-z_c|} = \frac{z_a-z_c}{|z_a-z_c|}e^{i\pi/2}$

Dividing both, we get

$$\frac{(z_a-z)|z_b-z_c|}{|z_a-z|(z_b-z_c)} = \frac{(z_c-z)|z_a-z_c|}{|z_c-z|(z_a-z_c)}$$

$\tan\alpha = \frac{|z_c-z|}{|z-z_a|} = \frac{|z_b-z_c|}{|z_c-z_a|}$ $\Rightarrow$ $\frac{(z_a-z)}{(z_b-z_c)}\times\frac{|z_b-z_c|^2}{|z_c-z_a|^2} = \frac{(z_c-z)}{(z_a-z_c)}$

$\Rightarrow (z_a-z)\left(\frac{\bar{z}_b-\bar{z}_c}{\bar{z}_a-\bar{z}_c}\right) = (z_c-z)$ $\Rightarrow$ $z = \frac{z_a(\bar{z}_b-\bar{z}_c)-z_c(\bar{z}_a-\bar{z}_c)}{(z_b-z_a)}$

**Ex.63** If complex number z lies on the curve $|z - (-1 + i)| = 1$, then find the locus of the complex number w $= \dfrac{z+i}{1-i}, i = \sqrt{-1}$.

**Sol.** $|z - (-1 + i)| = 1 \quad \Rightarrow \quad |z + 1 - i| = 1 \quad \ldots(1)$

Also $w = \dfrac{z+i}{1-i} \quad \Rightarrow \quad (1 - i)\, w = z + i \Rightarrow (1 - i)\, w - i = z$

$\Rightarrow |(1 - i)\, w - i + 1 - i| = |z + 1 - i| \Rightarrow |1 - i| \left| w + \dfrac{1-2i}{1-i} \right| = 1 \Rightarrow \left| w + \dfrac{(1-2i)(1+i)}{(1+i)(1-i)} \right| = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \left| w + \dfrac{3+i}{2} \right| = \dfrac{1}{\sqrt{2}} \Rightarrow \left| w - \dfrac{-3+i}{2} \right| = \dfrac{1}{\sqrt{2}} \Rightarrow$ locus of w is a circle centered at $\left(-\dfrac{3}{2}, \dfrac{1}{2}\right)$ and radius $\dfrac{1}{\sqrt{2}}$.

**(k) Reflection points for a straight line :** Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ . Note that the two points denoted by the complex numbers $z_1$ & $z_2$ will be the reflection points for the straight line $\bar{\alpha} z + \alpha \bar{z} + r = 0$ if and only if ; $\bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$, where r is real and $\alpha$ is non zero complex constant.

**(l) Inverse points w.r.t. a circle :** Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius $\rho$ , if :

**(i)** the point O, P, Q are collinear and on the same side of O. **(ii)** OP . OQ = $\rho^2$.

Note that the two points $z_1$ & $z_2$ will be the inverse points w.r.t. the circle

$z\bar{z} + \bar{\alpha} z + \alpha \bar{z} + r = 0$ if and only if $z_1 \bar{z}_2 + \bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$.

**Ex.64** A, B, C are the vertices of a triangle inscribed in the circle $|z| = 1$. Altitude from A meets the circle again at D. If D, B, C represents complex numbers $z_1$, $z_2$, $z_3$ respectively, then prove that the complex number representing the reflection of D in the line BC, is $\dfrac{z_1 z_2 + z_1 z_3 - z_2 z_3}{z_1}$.

**Sol.** If the reflection D in BC is $P(z_5)$, then $|z_2 - z_1| = |z_2 - z_5|$ and $|z_3 - z_1| = |z_3 - z_5|$.

The first relation is $(z_2 - z_1)(\bar{z}_2 - \bar{z}_1) = (z_2 - z_5)(\bar{z}_2 - \bar{z}_5)$

or $2 - z_1\bar{z}_2 - \bar{z}_1 z_2 = 1 - \bar{z}_2 z_5 - \bar{z}_5 z_2 + z_5 \bar{z}_5$ (since $z_1 \bar{z}_1 = 1 = z_2 \bar{z}_2$)

or $1 - \dfrac{z_1}{z_2} - \dfrac{z_2}{z_1} = z_5 \bar{z}_5 - \dfrac{z_5}{z_2} - \bar{z}_5 z_2$ or $z_1(z_2 + z_5) - (z_1^2 + z_2^2) = \bar{z}_5 (z_5 - z_2) z_1 z_2 \quad \ldots(1)$

Similarly, form the second relation, $z_1(z_3 + z_5) - (z_1^2 + z_3^2) = \bar{z}_5 (z_5 - z_3) z_1 z_3. \quad \ldots(2)$

Eliminating $\overline{z}_5$ from (1) and (2), we get

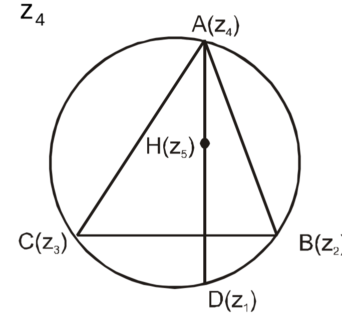
$z_3(z_5 - z_3)\,[z_1(z_2 + z_5) - (z_1^2 + z_2^2)] = z_2(z_5 - z_2)[z_1(z_3 + z_5) - ((z_1^2 + z_3^2)]$

or $z_5(z_2 = z_3)(z_1^2 - z_2z_3 - z_1z_5) = (z_2 - z_3)\,(z_1^2\,(z_2 + z_3) - z_1z_2z_3 - z_1z_5(z_2 + z_5))$

or $(z_5 - z_1)\,(-z_2z_3 + z_1(z_2 + z_3) - z_1z_5) = 0 \quad (z_2 \neq z_3)$ or $-z_2z_3 + z_1(z_2 + z_3) - z_1z_5 = 0\,(z_1 \neq z_5)$

Hence $z_5 = \dfrac{z_1z_2 + z_1z_3 - z_2z_3}{z_1}$

**Alternate :** Let H, G and O be respectively the orthocentre, the centroid and the circumcentre of $\Delta$ABC. Let A and H represent the complex numbers $z_4$ and $z_5$ $\Rightarrow \quad z_1 = -\dfrac{z_2z_3}{z_4}$




$\Rightarrow \quad z_4 = -\dfrac{z_2z_3}{z_1}$ and the complex number associated with the G is

$$\frac{z_1 + z_3 + z_4}{3} = \frac{z_2 + z_3 - \dfrac{z_2z_3}{z_1}}{3} = \frac{z_1z_2 + z_1z_3 - z_2z_3}{3z_1}.$$

Now the orthocentre is the reflection of D in the line BC, and G divides HO in the ratio 2 : 1. Since O represents the complex number zero, $\dfrac{z_1z_2 + z_1z_3 - z_2z_3}{3z_1} = \dfrac{z_5}{3} \Rightarrow z_5 = \dfrac{z_1z_2 + z_1z_3 - z_2z_3}{z_1}$.

## K. PTOLEMY'S THEOREM

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides. i.e. $|z_1 - z_3|\ |z_2 - z_4| + |z_1 - z_2|\ |z_3 - z_4| + |z_1 - z_4|\ |z_2 - z_3|$.

**Ex.65** If A, B, C and D represent the complex numbers $z_1$, $z_2$, $z_3$ and $z_4$, use the identity.
$(z_1 - z_4)\,(z_2 - z_3) + (z_2 - z_4)\,(z_3 - z_1) + (z_3 - z_4)\,(z_1 - z_2) = 0$
to show that AD . BC $\leq$ (BD . CA) + (CD . AB)

**Sol.** The given identity can be rewritten $(z_1 - z_4)\,(z_2 - z_3) = (z_4 - z_2)\,(z_3 - z_1) + (z_4 - z_3)\,(z_1 - z_2)$
$\Rightarrow \quad |(z_1 - z_4)\,(z_2 - z_3)| = |(z_4 - z_2)\,(z_3 - z_1) + (z_4 - z_3)\,(z_1 - z_2)|$
$\Rightarrow \quad |z_1 - z_4|\ |z_2 - z_3| \leq |z_4 - z_2|\ |z_3 - z_1| + |z_4 - z_3|\ |z_1 - z_2|$,
which proves the result, since
AD = $|z_1 - z_4|$, BD = $|z_4 - z_2|$, CD = $|z_3 - z_4|$, BD = $|z_2 - z_3|$,CA = $|z_3 - z_1|$, AB = $|z_1 - z_2|$.